Breakdown of the Landauer bound for information erasure in the quantum regime

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A known aspect of the Clausius inequality is that an equilibrium system subjected to a squeezing $dS<0$ of its entropy must release at least an amount $|đQ|=T|dS|$ of heat. This serves as a basis for the Landauer principle, which puts a lower bound $T\ln 2$ for the heat generated by erasure of one bit of information. Here we show that in the world of quantum entanglement this law is broken. A quantum Brownian particle interacting with its thermal bath can either generate less heat or even *absorb* heat during an analogous squeezing process, due to entanglement with the bath. The effect exists even for weak but fixed coupling with the bath, provided that temperature is low enough. This invalidates the Landauer bound in the quantum regime, and suggests that quantum carriers of information can be more efficient than assumed so far.

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I. INTRODUCTION

The laws of thermodynamics are at the basis of our understanding of nature, so it is rather natural that they have applications beyond their original scope, e.g., in computing and information processing [1–8]. The first connection between information storage and thermodynamics was made by von Neumann in the 1950s [2]. His speculation that each logical operation costs at least an amount of energy $T\ln 2$ was too pessimistic. Landauer pointed out that reversible ''one-to-one'' operations can be performed, in principle, without dissipation; only irreversible operations ''many-to-one'' operations, like erasure, require dissipation of energy, at an amount at least equal to the von Neumann estimate $T\ln 2$ per erased bit [4,5]. This conclusion is a direct consequence of the Clausius inequality, which connects the change of heat in a given process with the change of entropy. It is perfectly consistent with intuition, since as everybody had a chance to observe, equilibrium substances typically release heat under isothermal compression of their entropy (or volume, which is the same for a good majority of classical cases). Rather recently the basis of the effect was finally put on the Clausius inequality [8], and it was shown that the previous not very strict considerations are just particular cases of its application to information processing systems.

The principal importance of erasure among other information-processing operations originates from the fact that it is connected with changes of entropy, and thus cannot be realized in a closed system. One needs to couple the information-carrying system with its environment. Therefore, the process is accompanied with changes in heat whose magnitude has to be determined by thermodynamics. It was shown rigorously that all computations can be performed using reversible logical operations only [6].

Here we will consider thermodynamic aspects of erasure at low temperatures, so low that quantum effects start to play an important role. We choose the simplest working example: a one-dimensional Brownian particle in contact with a thermal bath at temperature T, subject to an external confining potential. The main new aspect arising at low temperatures is entanglement of the Brownian particle with the bath. Therefore, even when the total system is in a pure state, the subsystem (the Brownian particle) is in a mixed state. Thus its stationary state cannot be given by equilibrium (Gibbsian) quantum thermodynamics, which would predict for $T\to 0$ that the subsystem goes to its ground state, the pure vacuum state. The latter can only be approached for not too low temperatures, given a fixed but weak coupling with the bath. In general, a new situation arises, for which a generalized thermodynamical interpretation can be given [9,10] (for analogous situations in glasses and related systems see Refs. [11,12]). In particular, the classical Clausius inequality is invalid. We stress that this situation is not at all exceptional, since it appears even for a small but generic coupling provided that temperature is low enough.

Our main result will show that when entropy of the particle is decreased by external agents, namely, when a part of the information carried by it is erased, the particle can absorb heat in clear contrast with the classical intuition. Later we shall apply this result to show that there is not anything similar to Landauer bound at low temperatures. Thus in this respect quantum carriers of information can be more efficient than their classical analogs.

Since we are in a new situation where relations of the standard thermodynamics are possibly broken, we prefer to work with simple exactly-solvable models, where all general relations can be illustrated or disproved explicitly. In analogous situation with the classical theory Szilard [3] used a model with one classical Brownian particle interacting with its thermal bath.

This paper is organized as follows. In Sec. II we review the connection between thermodynamics and information erasure. Section III is devoted to heat and entropy changes of a quantum Brownian particle in contact with its thermal bath. This model can be considered as an extension to the quantum regime of the seminal model of Szilard [3]. In Sec. IV our main results on violation of the Landauer principle are presented. In Sec. V we analyze the most popular derivation of this principle [4–6], in order to show where its arguments appear to be inapplicable. Our conclusions are presented in Sec. VI.

II. ERASURE OF INFORMATION AND GIBBSIAN THERMODYNAMICS

A. Source of information

Let us start by briefly recalling what is meant by erasure of information. Since information is carried by physical systems, messages are coded by their states, namely, every state (or possibly group of states) corresponds to a ''letter.'' The simplest example is a two-state system, which carries on one bit of information. The basic model of *source of information* in Shannonean, probabilistic information theory [13–15] assumes that the carrier of information can be in different states with certain (so called *a priori*) probabilities. In other words, the messages of this source appear randomly and the measure of their expectation is given by the corresponding probabilities. For example, in the classical case the carrier of information may occupy a cell in its phase space with volume $dx\,dp/(2\pi\hbar)$ with *a priori* probability $P(x,p)$. Then different cells will correspond to different messages. In the quantum case the completely analogous situation is described by a density matrix ρ,

$$\rho=\sum_n p_n|n\rangle\langle n|, \tag{2.1}$$

$$\langle n|m\rangle=\delta_{nm}, \tag{2.2}$$

which means that the carrier occupies a state $|n\rangle$ with the *a priori* probability p_n. Moreover, different quantum states are exclusive as indicated by Eq. (2.2). As in the classical case, the appearance of the carrier in different states will bring different messages.

The fundamental theorem by Shannon [13–15] states that the information carried by an information source is given by its entropy. Namely, it is equal to

$$S=-\int\frac{dx\,dp}{2\pi\hbar}P(p,x)\ln P(p,x), \tag{2.3}$$

in the classical case, and to

$$S_{\mathrm{vN}}(\rho)=-\sum_n p_n\ln p_n=-\mathrm{tr}(\rho\ln\rho), \tag{2.4}$$

in the quantum situation. Here $S_{\mathrm{vN}}(\rho)$ is the von Neumann entropy of the density matrix ρ. The physical meaning of this result can be understood as follows. A source that has lower entropy occupies fewer states with higher probability. It can be said to be better known, and, therefore, the appearance of its results will bring less information. In contrast, a source with higher entropy occupies more states with lower probability. Its messages are less expectable, and, therefore, bring more information. The rigorous realization of this intuitive argument appeared to be the most straightforward and fruitful proof of the Shannon theorem [15,18]. Notice that the entropies (2.3) and (2.4) appear here on the information theoretical footing and not as purely thermodynamical quantities [15].

The above notion of an information source does not exhaust the full meaning of this concept. Here it appears as a model of the probabilistic information theory. Advantages and shortcomings of this approach were nicely reviewed by Kolmogorov [16].

B. Erasure

Erasure is an operation that is done by an external agent in order to reduce the entropy of the information carrier. This means that in its final state the carrier brings less information, i.e., some amount of it has been erased. In particular, a complete erasure corresponds to the minimization of entropy. Notice that erasure is defined as a ''blind'' operation, which is done independently on the actual state of the information carrier. This is how information processing systems operate: they do not recognize the actual state of a bit before erasing it. Following standard assumptions [17,18] we will model external operations by a time-dependent Hamiltonian $H(t)$ of the carrier, namely, some of its parameters will be varied with time according to given trajectories. If the information carrying system is closed, then its dynamics is described by the Liouville equation for $P(x,p)$ in the classical case or by the von Neumann equation

$$\frac{d}{dt}\rho=\frac{i}{\hbar}[\rho(t)H(t)-H(t)\rho(t)], \tag{2.5}$$

in the quantum situation. As can be shown directly, the entropies (2.3) and (2.4) remain constant in time. In order to change them, one has to consider an information carrier, which is an open system. In that case a part of its energy will be controlled (i.e., transferred or received) by its environment as *heat*. Indeed, if

$$U=\mathrm{tr}[H(t)\rho(t)] \tag{2.6}$$

is the average energy of the carrier, then its change during a time dt reads

$$dU=đ\mathcal{Q}+đ\mathcal{W}=\mathrm{tr}[H\,d\rho]+\mathrm{tr}[\rho\,dH]. \tag{2.7}$$

This is the energetic budget of the system. The last term represents the averaged mechanical work $đ\mathcal{W}$ produced by an external agent [17,18]. The first term in right-hand side of Eq. (2.7) arises due to the statistical redistribution in phase space. We shall identify it with the change of heat $đ\mathcal{Q}$ [17,18], so Eq. (2.7) is just the first law. As can be shown through Eq. (2.5), the heat is explicitly zero for a closed system. All these formulas are valid in the classical case as well. Here ρ should be substituted by $P(x,p)$, and the trace will be changed by the integration over the corresponding phase space.

C. Brownian particle as an information carrier

In order to specify the situation, let us consider a Brownian particle as an information carrying system. A similar simple model was employed by Szilard [3] in his seminal analysis of the Maxwell's demon problem. The Brownian particle has a Hamiltonian $H(p,x,t)$, where p,x are coordinate and momentum. A parameter that varies with time can be the mass of the particle or the shape of its potential energy. The environment of the particle will be taken to be a thermal bath. This is a generic situation, in the sense that the bath satisfies the following generally accepted conditions [17–19], which are identical for quantum and classical situations:

(1) The interaction between the particle and bath is linear. It is assumed to be so weak that the nonlinear modes of the bath are not excited, and the bath itself can be modeled as a collection of harmonic oscillators [10,20,19]. This assumption has been verified rigorously, when starting from rather general microscopic situations.

(2) The bath is a macroscopic system; the thermodynamic limit has been taken for it.

(3) Before it started to interact with the particle at some initial time, the bath was in thermal equilibrium (i.e., in a Gibbsian state) at temperature *T*. This temperature will be refered to as the temperature of the bath. This assumption reflects the typical macroscopic preparation at the initial time.

(4) The particle and bath together form a closed system. Thus, the overall system is described by the Schrödinger equation (alternatively Heisenberg equations) in the quantum case, and by Newton's equations in the classical case.

A minimal model, which incorporates all these properties was proposed in Refs. [21,22], and later became known as the Caldeira-Leggett model [20,19]. The above assumptions ensure that the reduced dynamics of the Brownian particle will be given by the quantum or classical Langevin equations [19].

As a result of interaction with the macroscopic bath, the Brownian particle will relax with time towards a definite stationary state. In the present paper we will additionally assume that all external operations on the particle are adiabatic, namely, they occur on time-scales that are much larger than the characteristic relaxation time. There are several physical reasons for this restriction. First, in many circumstances the adiabatic process can be shown to be optimal, in the sense that it is connected with minimal amount of work done by the external agent [10,17,18]. On the other hand, this time-scale separation more naturally corresponds to the interaction between a deterministic agent and the microscopic particle.

Let us now consider the classical and quantum situation separately.

1. Classical case

As it is well known, under the above standard assumptions on the thermal bath the classical Brownian particle relaxes to the Gibbs distribution [17,18,23]

$$P(p,x)=\frac{1}{Z}\exp\left[-\frac{1}{T}H(p,x)\right],$$

$$Z=\int dx\,dp\,\exp\left[-\frac{1}{T}H(p,x)\right]. \tag{2.8}$$

Since the external operation is assumed to be adiabatic, the time-dependent distribution of the particle will be given by Eq. (2.8) with the corresponding time-dependent Hamiltonian $H(x,p,t)$. The Clausius equality

$$đ\mathcal{Q}=TdS, \tag{2.9}$$

which connects the changes of heat and entropy during the process can be derived directly from Eqs. (2.6), (2.3), and (2.7). It holds that when compressing the phase space of the particle ($dS<0$), it releases heat ($đ\mathcal{Q}<0$). Since for any nonadiabatic change one has $đ\mathcal{Q}\leqslant TdS$ (Clausius inequality), $|đ\mathcal{Q}|$ can only be larger if the process is not very slow. In other words, the minimal amount of the released heat is equal to $|T\,dS|$. This is the Landauer principle.

2. Quantum case

Let us now move to the quantum domain, which in the present context just means the domain of low temperatures. We *assume* that the quantum carrier of information interacts with its thermal bath, but so weakly, that it is described by quantum Gibbs distribution at the bath temperature *T*,

$$\rho=\frac{1}{Z}\exp\left[-\frac{H}{T}\right],\quad Z=\mathrm{tr}\exp\left[-\frac{H}{T}\right]. \tag{2.10}$$

The concrete conditions on the weakness of the interaction will be discussed later. It can now easily be seen that, provided we use entropy as defined in Eq. (2.4), the Clausius inequality (2.9) still holds and all its consequences including the Landauer principle are generalized automatically. The important difference between the classical and quantum cases has to be noted already here: In the classical situation there is no limitation on the interaction strength, and the classical Gibbs distribution appears naturally from the above standard conditions on the thermal bath.

III. QUANTUM BROWNIAN PARTICLE IN CONTACT WITH ITS THERMAL BATH

A. Wigner function and effective temperatures

As explained in Sec. I, at low temperatures of the bath the Brownian particle is not described by the quantum Gibbs distribution, except for very weak interaction with the bath. Therefore, its state at low temperatures has to be found from first principles, starting from the microscopic description of the bath and the particle. This program was realized in Refs. [9,10,19,24]. In particular, in Refs. [9,10] we investigated statistical thermodynamics of the quantum Brownian particle.

Here we consider the simplest example, a harmonic oscillator with Hamiltonian

$$H(p,x)=\frac{p^2}{2m}+\frac{ax^2}{2}, \tag{3.1}$$

where m is the mass, and a is the width. The state of this particle can be described through the Wigner function [18]. Recall that in quantum theory this object plays nearly the same role as the common distribution of coordinate and momentum in the classical theory. The stationary Wigner function reads [25,19,9,10]

$$W(p,x)=W(p)W(x)=\frac{1}{2\pi}\sqrt{\frac{a}{mT_pT_x}}\exp\left[-\frac{p^2}{2mT_p}-\frac{ax^2}{2T_x}\right], \tag{3.2}$$

where $W(p)$, $W(x)$ are the probability distributions of momenta and coordinate, and where

$$T_x=a\langle x^2\rangle, \quad T_p=\frac{\langle p^2\rangle}{m} \tag{3.3}$$

are two effective temperatures, to be discussed a bit later. Equation (3.2) represents the state of the particle, provided that the interaction with the bath was switched on long time before, so that the particle already came to its stationary state. The effective temperatures T_p and T_x depend not only on the system parameters m, T, and a, but also on the damping constant γ, which quantifies the interaction with the thermal bath, and on a large parameter Γ that is the maximal characteristic frequency of the bath. In particular, the Gibbsian limit corresponds to $\gamma\to 0$. Then the distribution (3.2) tends to the quantum Gibbsian: $T_x=T_p=\frac{1}{2}\hbar\omega_0\coth(\frac{1}{2}\beta\hbar\omega_0)$, where $\omega_0=\sqrt{a/m}$ [see Eqs. (3.16) and (3.17)]. In the classical limit, which is realized for $\hbar\to 0$ or $T\to\infty$, the dependence on γ and Γ disappears; both T_p and T_x go to T, reproducing the classical Gibbsian distribution (2.8). The appearance of the effective temperatures in the quantum regime can be understood as follows. For $T\to 0$ quantum Gibbs distribution predicts the pure vacuum state for the particle. Due to quantum entanglement this cannot be the case for a nonweakly interacting particle, so must T_x, T_p depend on γ, and, being nontrivial, they have to be obtained from first principles, as the state is not Gibbsian. The exact expressions for T_p, T_x reads [10,25,19]

$$T_p=\frac{\hbar\gamma\Gamma^2}{\pi m}\left[\frac{\omega_1^2\psi_1}{(\omega_2^2-\omega_1^2)(\omega_3^2-\omega_1^2)}+\frac{\omega_2^2\psi_2}{(\omega_1^2-\omega_2^2)(\omega_3^2-\omega_2^2)}+\frac{\omega_3^2\psi_3}{(\omega_1^2-\omega_3^2)(\omega_2^2-\omega_3^2)}\right]-T, \tag{3.4}$$

$$T_x=-\frac{a\hbar\gamma\Gamma^2}{m^2\pi}\left[\frac{\psi_1}{(\omega_2^2-\omega_1^2)(\omega_3^2-\omega_1^2)}+\frac{\psi_2}{(\omega_1^2-\omega_2^2)(\omega_3^2-\omega_2^2)}+\frac{\psi_3}{(\omega_1^2-\omega_3^2)(\omega_2^2-\omega_3^2)}\right]-T, \tag{3.5}$$

where $\psi_k=\psi(\hbar\beta\omega_k/2\pi)$ for $k=1,2,3$ and $\beta=1/T$ as usual. Further $\psi(z)=\Gamma'(z)/\Gamma(z)$ is Euler's ψ function. $\omega_{1,2,3}$ are roots of the following cubic equation:

$$(\Gamma-\omega)(\omega^2+\omega_0^2)-\omega\frac{\gamma\Gamma}{m}=0. \tag{3.6}$$

In the present paper we will be mostly interested in the so-called quasi-Ohmic limit where Γ is the largest characteristic frequency of the problem. This is the most realistic situation for information storing devices. In this limit one approximately has

$$\omega_{1,2}=\frac{\gamma}{2m}(1\pm\sqrt{1-4\xi})+\frac{\gamma^2}{2\Gamma m^2}\left[1\pm\frac{1-2\xi}{\sqrt{1-4\xi}}\right], \tag{3.7}$$

$$\omega_3=\Gamma-\frac{\gamma}{m}-\frac{1}{\Gamma}\left(\frac{\gamma}{m}\right)^2, \tag{3.8}$$

where $\xi=am/\gamma^2$ characterizes the relative importance of damping: $\xi\ll 1$ corresponds to overdamped motion, while $\xi\gg 1$ indicates underdamping. Since Γ is large, we need the first leading terms in Eqs. (3.7) and (3.8). This brings [10]

$$T_p=\frac{\hbar}{\pi(\omega_1-\omega_2)}\left[(\omega_1^2-\omega_2^2)\psi\left(\frac{\beta\hbar\Gamma}{2\pi}\right)-\omega_1^2\psi\left(\frac{\beta\hbar\omega_1}{2\pi}\right)+\omega_2^2\psi\left(\frac{\beta\hbar\omega_2}{2\pi}\right)\right]-T, \tag{3.9}$$

$$T_x=\frac{\hbar a}{m\pi(\omega_1-\omega_2)}\left[\psi\left(\frac{\beta\hbar\omega_1}{2\pi}\right)-\psi\left(\frac{\beta\hbar\omega_2}{2\pi}\right)\right]-T. \tag{3.10}$$

Limiting cases can be studied with help of the following approximate values for the ψ function:

$$\psi(x)=-\frac{1}{x}-\gamma_E+x\frac{\pi^2}{6}, \quad |x|\ll 1, \tag{3.11}$$

$$\psi(x)=\ln x-\frac{1}{2x}-\frac{1}{12x^2}, \quad |x|\gg 1, \tag{3.12}$$

where x is a complex number, and $\gamma_E=0.577\,216$ is the Euler constant. In the low-temperature limit $T\to 0$ we obtain from Eqs. (3.9) and (3.10)

$$T_p=\frac{\hbar[\omega_1^2\ln(\Gamma/\omega_1)-\omega_2^2\ln(\Gamma/\omega_2)]}{\pi(\omega_1-\omega_2)}+O(T^4), \tag{3.13}$$

$$T_x = \frac{\hbar a}{\pi m(\omega_1 - \omega_2)} \ln \frac{\omega_1}{\omega_2} + \frac{\pi\gamma}{3\hbar a} T^2 + O(T^4). \quad (3.14)$$

The weak-coupling limit can be obtained from Eqs. (3.10) and (3.9) taking $\xi \gg 1$ and noticing that

$$\frac{\psi(ix) - \psi(-ix)}{i\pi} = \frac{1}{x\pi} + \coth(x\pi), \quad (3.15)$$

which is obtained from the reflection formula: $\Gamma(z)\Gamma(1-z) = \pi/(\sin \pi z)$. Here one has the following expressions:

$$T_p = T_x + \frac{\hbar\gamma}{4\pi m}\left[2\psi\left(\frac{\beta\hbar\Gamma}{2\pi}\right) - \psi\left(\frac{i\beta\hbar\omega_0}{2\pi}\right) - \psi\left(-\frac{i\beta\hbar\omega_0}{2\pi}\right)\right], \quad (3.16)$$

$$T_x = \frac{\hbar\omega_0}{2}\coth\frac{\hbar\beta\omega_0}{2} + \frac{\hbar\gamma}{4\pi m}\mathcal{G}\left(\frac{\hbar\beta\omega_0}{2}\right), \quad (3.17)$$

$$\mathcal{G}(x) = ix[\psi'(-ix) - \psi'(ix)]. \quad (3.18)$$

The asymptotic expressions of these quantities read in the opposite, strongly damped region $\xi \ll 1$ and for low T,

$$T_p = \frac{\hbar\gamma}{\pi m}\ln\frac{\Gamma m}{\gamma} + \frac{\hbar a}{\pi\gamma} + O(T^4), \quad (3.19)$$

$$T_x = \frac{\hbar a}{\pi\gamma}\ln\frac{\gamma^2}{am} + \frac{\pi\gamma}{3\hbar a}T^2 + O(T^4). \quad (3.20)$$

It is interesting to mention as well the high-temperature (quasiclassical) asymptotic values for T_p, T_x. Applying Eq. (3.11) one gets

$$T_p = T + \frac{\hbar^2(am - \gamma^2 + \Gamma m\gamma)}{12m^2 T} + O(\hbar^3\beta^2), \quad (3.21)$$

$$T_x = T + \frac{\hbar^2 a}{12mT} + O(\hbar^3\beta^2). \quad (3.22)$$

B. Energy and partial entropies

The average energy of the Brownian particle

$$U = \int dx\, dp W(p,x) H(p,x) = \frac{T_p}{2} + \frac{T_x}{2} \quad (3.23)$$

depends on a and m, in contrast to its classical value T. We will need entropies of momentum and coordinate distribution

$$S_p = -\int dp\; W(p) \ln W(p) = \frac{1}{2}\ln(mT_p), \quad (3.24)$$

$$S_x = -\int dx\; W(x) \ln W(x) = \frac{1}{2}\ln\frac{T_x}{a}. \quad (3.25)$$

The ‘‘Boltzmann’’ entropy reads

$$S_B = -\int dp\, dx\; W(p,x) \ln[\hbar W(p,x)] = S_p + S_x - \ln\hbar$$
$$= \frac{1}{2}\ln\frac{mT_pT_x}{a\hbar^2}. \quad (3.26)$$

Notice that they all are different from $S_{\mathrm{vN}}(\rho)$ defined by Eq. (2.4).

C. Heat and work

The expressions for heat and work are generalized from Eqs. (2.7) by simply using the Wigner function $W(p,x)$ instead of ρ. This can be easily verified, when using Eq. (3.37). One can prove by a direct calculation that quantities T_p, T_x do deserve their nomenclature, since the classical Clausius equality can be generalized as

$$dU = đ\mathcal{Q} + đ\mathcal{W} = T_p\, dS_p + T_x\, dS_x + đ\mathcal{W}, \quad (3.27)$$

for variation of any parameter. We will be especially interested in variation of the mass and the width of the potential. The corresponding changes of heat read

$$đ_a\mathcal{Q} = \frac{1}{2}\left(\frac{\partial T_p}{\partial a} + \frac{\partial T_x}{\partial a} - \frac{T_x}{a}\right) da, \quad (3.28)$$

$$đ_m\mathcal{Q} = \frac{1}{2}\left(\frac{\partial T_p}{\partial m} + \frac{\partial T_x}{\partial m} + \frac{T_p}{m}\right) dm. \quad (3.29)$$

Using Eqs. (3.10)–(3.12) one gets expressions for heat. Let us introduce the following notations:

$$\alpha_1 = \frac{\hbar\gamma}{4\pi mT}, \quad \alpha_2 = \frac{a\hbar}{\pi\gamma T}, \quad (3.30)$$

and then derive

$$\frac{\partial\mathcal{Q}}{\partial a} = -\frac{T}{2a} + \frac{\hbar^2}{24mT}, \quad \alpha_1 \ll 1, \quad (3.31)$$

$$\frac{\partial\mathcal{Q}}{\partial a} = -\frac{\pi\gamma T^2}{3\hbar a^2}, \quad \alpha_1 \gg 1, \quad (3.32)$$

$$\frac{\partial\mathcal{Q}}{\partial m} = \frac{T}{2m} + \frac{\hbar^2\gamma^2(z^2+1)}{66m^3T}, \quad \alpha_2 \ll 1, \quad (3.33)$$

$$\frac{\partial\mathcal{Q}}{\partial m} = \frac{\hbar\gamma}{2\pi m^2}, \quad \alpha_2 \gg 1. \quad (3.34)$$

Notice that the last equation applies not only for low temperatures, but also for weak coupling [see Eq. (3.30)]. Using these results one can show that

$$\frac{\partial\mathcal{Q}}{\partial a} \leq 0, \quad \frac{\partial\mathcal{Q}}{\partial m} \geq 0, \quad (3.35)$$

for all values of parameters including, of course, the classical limit. For the work done in this process one obtains in a simple manner from the Hamiltonian (3.1) and Eq. (3.3)

$$\frac{\partial\mathcal{W}}{\partial a}=\left\langle\frac{\partial H}{\partial a}\right\rangle=\frac{1}{2}\langle x^2\rangle=\frac{1}{2}\,\frac{T_x}{a}\geqslant 0,$$

$$\frac{\partial\mathcal{W}}{\partial m}=\left\langle\frac{\partial H}{\partial m}\right\rangle=-\frac{1}{2}\,\frac{\langle p^2\rangle}{m^2}=-\frac{1}{2}\,\frac{T_p}{m}\leqslant 0. \tag{3.36}$$

It is interesting to note that the signs of $\partial\mathcal{Q}$ and $\partial\mathcal{W}$ in Eqs. (3.35) and (3.36) are the same as in the classical case, where $T_x=T_p=T$.

D. Density matrix

To investigate the von Neumann entropy (2.4) one needs the density matrix corresponding to the Wigner function (3.2). Applying the standard relation between the density matrix in coordinate representation and the Wigner function,

$$\left\langle x+\frac{u}{2}|\rho|x-\frac{u}{2}\right\rangle=\int dp\, e^{-ipu/\hbar}W(p,x), \tag{3.37}$$

one gets the following expression:

$$\langle x|\rho|x'\rangle=\frac{1}{\sqrt{2\pi\langle x^2\rangle}}\exp\left[-\frac{(x+x')^2}{8\langle x^2\rangle}-\frac{(x-x')^2}{2\hbar^2/\langle p^2\rangle}\right]. \tag{3.38}$$

The physical meaning of Eq. (3.38) is clear: The diagonal elements ($x=x'$) are distributed at the scale $\sqrt{\langle x^2\rangle}$, while the maximally off-diagonal elements ($x=-x'$), which characterize coherence, are distributed with the characteristic scale $\hbar/\sqrt{\langle p^2\rangle}$. We have to find eigenfunctions and eigenvectors of this density matrix, $\int dx'\langle x|\rho|x'\rangle f_n(x')=p_n f_n(x)$. The solution of this problem involves some tabulated formulas for Hermite polynoms, and results in

$$p_n=\frac{1}{w+\frac{1}{2}}\left[\frac{w-\frac{1}{2}}{w+\frac{1}{2}}\right]^n, \tag{3.39}$$

$$f_n(x)=c\,H_n(c\,x)e^{-c^2x^2/2},\quad c=\left(\frac{\langle p^2\rangle}{\hbar^2\langle x^2\rangle}\right)^{1/4}, \tag{3.40}$$

$$w=\frac{\Delta p\,\Delta x}{\hbar}=\sqrt{\frac{\langle p^2\rangle\langle x^2\rangle}{\hbar^2}}=\sqrt{\frac{mT_pT_x}{\hbar^2 a}}, \tag{3.41}$$

where H_n are Hermite polynomials, and it holds that $w\geqslant\frac{1}{2}$ due to the Heisenberg uncertainty relation. The result for the von Neumann entropy (2.4) now reads [19]

$$S_{vN}=(w+\tfrac{1}{2})\ln(w+\tfrac{1}{2})-(w-\tfrac{1}{2})\ln(w-\tfrac{1}{2}). \tag{3.42}$$

The first terms in its large w expansion read

$$S_{vN}=\ln w+1-\frac{1}{24w^2}-\frac{1}{320w^4}-\frac{1}{2688w^6}. \tag{3.43}$$

Notice that the same quantity w governs the Boltzmann entropy

$$S_B=S_p+S_x-\ln\hbar=\ln w+1. \tag{3.44}$$

This appears to coincide with the leading terms of Eq. (3.43). It is known to be larger than the von Neumann entropy, and this is obvious from the sign of the correction terms in Eq. (3.43).

If some parameter (a or m) is varied, then the derivative of S_{vN} with respect to it reads

$$dS_{vN}=\ln\frac{w+\frac{1}{2}}{w-\frac{1}{2}}\,dw. \tag{3.45}$$

In other words, the sign of the change in S_{vN} is determined by the sign of the change in w. This holds as well for the change in S_B, so qualitatively they carry the same information.

In this context let us stress again that von Neumann entropy $S_{\mathrm{vN}}(\rho)$ is the unique quantum measure of localization and information, whereas the entropies S_p, S_x characterize localizations of momenta and coordinate separately. Differences between S_p+S_x and S_{vN} are due to the fact that in quantum theory momentum and coordinate cannot be measured simultaneously; in this sense S_p+S_x characterize two different measurement setups. Nevertheless, for the harmonic particle if S_{vN} increases (decreases), then S_p+S_x increases (decreases) as well. Notice that the real importance of S_p, S_x becomes clear when they have to be used to generalize the Clausius inequality. The von Neumann entropy S_{vN} cannot be used for this purpose if $T_x\neq T_p$, i.e., when $\gamma\neq 0$.

IV. ENTROPY DECREASE WITH HEAT ABSORPTION

Now we will show that there are erasure processes, namely, processes where $dS_{\mathrm{vN}}\leqslant 0$, which are accompanied by an absorption of heat. We noticed already that heat is always absorbed, when the mass is increased [see Eq. (3.34)]. It will now be shown that there is a mass-increasing process, where $dS_{\mathrm{vN}}\leqslant 0$. Using Eqs. (3.20) and (3.19) one has at very low temperatures

$$\frac{\partial w^2}{\partial m}=\frac{\partial}{\partial m}\left[\frac{1}{\hbar^2}\langle x^2\rangle\langle p^2\rangle\right]=\frac{a}{\pi^2\gamma^2}\left[-1-\frac{\gamma^2}{am}\ln\frac{\Gamma m}{\gamma}+\left(1+\frac{\gamma^2}{am}\right)\ln\frac{\gamma^2}{am}\right]. \tag{4.1}$$

This expression is negative in its range of applicability.

An analogous argument can be brought about in the weak-coupling case. Starting from Eqs. (3.13) and (3.14) or alternatively from Eqs. (3.16) and (3.17) one derives the fol-

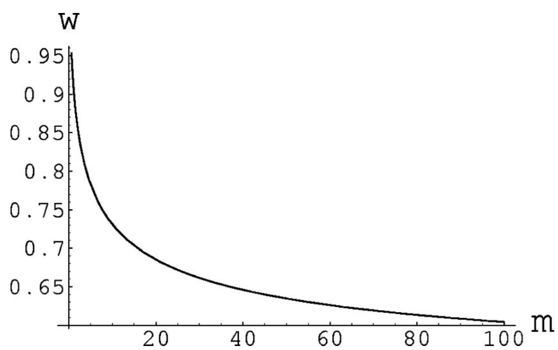

FIG. 1. Dimensionless phase space volume $w = \Delta p \Delta x/\hbar = \sqrt{\langle x^2\rangle\langle p^2\rangle/\hbar^2}$ vs mass m. The other parameters are $a=\gamma=1$, $\Gamma=500$, $\hbar=1$, and $T=0$. It is seen that the volume decays monotonically towards its minimal value 1/2, set by the uncertainty relation.

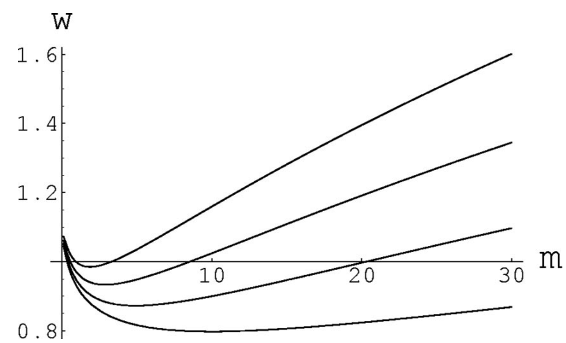

FIG. 2. Dimensionless phase space volume $w = \Delta p \Delta x/\hbar = \sqrt{\langle x^2\rangle\langle p^2\rangle/\hbar^2}$ vs mass m. From top to the bottom: $T=0.25$, 0.20, 0.15, and 0.10. The other parameters are the same as in Fig. 1: $a=\gamma=1$, $\Gamma=500$, and $\hbar=1$. It is seen that region of m's where the volume decays. This region is completely shrunk for $T=0.475\,53$. For higher temperatures the phase space volume monotonically increases with m.

lowing expressions for the effective temperatures in the weak-coupling $\gamma\to 0$ and low-temperature limit

$$T_p = \frac{\hbar\omega_0}{2} + \frac{\hbar\gamma}{\pi m}\ln\frac{\Gamma}{\omega_0\sqrt{e}}, \quad T_x = \frac{\hbar\omega_0}{2} - \frac{\hbar\gamma}{2\pi m}. \tag{4.2}$$

This implies

$$\frac{\partial w^2}{\partial m} = -\frac{\gamma}{4m\sqrt{am}}\ln\frac{\Gamma}{\omega_0 e^2}, \tag{4.3}$$

which is again negative in its range of applicability $\Gamma \gg \omega_0$. The general situation at low temperatures is illustrated by Fig. 1, where it is seen that w monotonically decreases when increasing the mass. In the limit $m\to\infty$ it tends to its corresponding Gibbsian value. This can be understood by noticing that the stationary state of a very heavy Brownian particle will not be influenced much by the bath. Indeed, as seen from Eqs. (3.7)–(3.10) the dimensionless parameter that controls transition from the weakly damped to the strongly-damped regime is $\xi = am/\gamma^2$. So to increase the mass while all other parameters are kept fixed, produces the same effect as to decrease the coupling constant γ.

Recall that the corresponding expression for $\partial\mathcal{Q}/\partial m$ was positive. This just means that for the variation of m we have an interesting case where heat is absorbed when entropy is decreasing. This is a counterexample for the general validity of the Landauer principle.

A. Where classical intuition is correct and where it fails

In the classical case one has an intuitively clear result: Upon increasing the mass of the particle it absorbs heat, $d\!\!\!^{-}\mathcal{Q} = dm\,\partial\mathcal{Q}/\partial m > 0$, while it performs work against the external agent, $d\!\!\!^{-}\mathcal{W} = -d\!\!\!^{-}\mathcal{Q} < 0$. In doing so, its entropy increases, $dS = d\!\!\!^{-}\mathcal{Q}/T$.

In the quantum case heat is also released and work is also done on the environment, as follows from Eq. (3.36). However, an unusual point appears: The quantum particle decreases its entropy when the mass is increased, since $\partial S_{\rm vN}/\partial m < 0$. To understand this point we notice that at low temperatures of the bath the particle has an appreciable entropy due to its entanglement with the bath [recall that for zero temperature the state of the overall system (particle plus bath) is pure, and the von Neumann entropy of the particle is the adequate measure of its entanglement with the bath]. When its mass is increased, its state moves towards the Gibbsian limit, and the entropy is reduced just because in the zero-temperature Gibbsian case the entropy is exactly zero. This decrease will also occur for low but finite temperatures of the bath, when entanglement still contributes to the entropy. So it is *quantum entanglement* that necessarily leads to this counterintuitive result.

Notice that this effect does not imply a violation of the second law in Thomson's formulation, which speaks about the impossibility to extract work by a cyclic variation of a system parameter [9,10]. Indeed, if after increasing the mass, one decreases it in order to complete the cycle, the external agent will do work on the particle, and it will release heat, thereby nullifying the overall work and heat (as expected, the overall work is positive if nonadiabatic variations are considered [9,10]).

B. Finite temperatures

The above effect $\partial w/\partial m < 0$ was analytically illustrated for $T\to 0$. However, it persists as well at finite, but not too large temperatures. This situation is illustrated in Fig. 2. Since in the classical case, namely, with high temperatures, one always has $\partial_m w = T/(2\sqrt{ma}) > 0$, we expect that the region with $\partial_m w < 0$ will completely disappear at some finite critical temperature. This is indeed the case, as Fig. 2 shows.

C. Variation of the spring constant

The analogous variation of a does not lead to such an unusual result. Here instead of Eq. (4.1) one has

$$\frac{\partial}{\partial a}\left[\frac{1}{\hbar^2}\langle x^2\rangle\langle p^2\rangle\right] = \frac{m}{\pi\hbar\gamma^2}\left[-1 - \frac{\gamma^2}{am}\ln\frac{\Gamma m}{\gamma} + \ln\frac{\gamma^2}{am}\right] \leqslant 0, \tag{4.4}$$

but the corresponding expression (3.28) for $\partial \mathcal{Q}/\partial a$ is negative as well. An expression analogous to Eq. (4.1) can be obtained for the weak-coupling limit. Using Eq. (4.2) one gets

$$\frac{\partial w^2}{\partial a}=-\frac{\gamma}{a\sqrt{am}}\ln\frac{\Gamma}{\omega_0}<0. \tag{4.5}$$

An important fact should be mentioned here. Although the particle just releases heat during localization, this heat is smaller than expected, since it scales at low temperatures as $|đ\mathcal{Q}|\sim T^2\,da$ [see Eq. (3.32)]. This already invalidates the Landauer bound $|đ\mathcal{Q}|\geqslant T|dS|\sim T\,da$, now only in magnitude, and not in sign. In this context the parameters of a statistical system can be distinguished as active and passive. In our concrete case the width of the potential a is a passive parameter, in a sense that its variations result in effects, which for any temperature are qualitatively (but not quantitatively) similar to the classical case. In contrast, the active parameters (in our case it is the mass m) invert their behavior at low temperatures.

D. Weak-coupling limit

Here we will especially point out on applicability of our result in the weak-coupling limit. First we will make an obvious remark that the precise meaning of this limit *must not* be understood in the sense $\gamma\equiv 0$, since the damping constant γ is never explicitly zero in practice, and having put it zero one will not have at all a possibility to change the entropy of the particle. The weak-coupling limit is understood in a sense that the interaction energy of the particle and the bath happened to be sufficiently small compared to the energy of the particle itself [18] (since the energy of the bath is infinite there is no need to involve the bath here). For low temperatures and $\gamma\to 0$ the energy of the particle is given by its zero-point value $\frac{1}{2}\hbar\sqrt{a/m}$. Because the interaction energy is explicitly zero for $\gamma\equiv 0$, it will be enough to choose γ sufficiently small to ensure the above condition of the weak-coupling limit.

Let us now turn to Eq. (3.34) that represents the amount of heat obtained by the particle when changing the mass at low temperatures. It is seen that in the leading order this quantity is proportional to γ. Thus, although the particle is in the weak-coupling regime, it still gets a positive (though small) amount of heat during the enhancement of its mass.

E. High temperatures

Finally we wish to show in more detail that the Landauer principle does hold in our model for sufficiently high temperatures. This follows from the fact that in this limit $T_p,T_x\to T$ as seen from Eqs. (3.21) and (3.22). A more elaborated discussion goes as follows. One has the following exact relation:

$$đ\mathcal{Q}-T\,dS_{\mathrm{vN}}=(T_x-T)dS_x+(T_p-T)dS_p -Td(S_{\mathrm{vN}}-S_x-S_p). \tag{4.6}$$

One applies here Eqs. (3.21), (3.22), and (3.43) to get for variation of m,

$$\frac{\partial \mathcal{Q}}{\partial m}-T\frac{\partial S_{\mathrm{vN}}}{\partial m}=-\frac{\hbar^2\gamma^2}{24m^3T}+O(\hbar^3\beta^2). \tag{4.7}$$

The deviation from the Clausius equality, and thus from the Landauer principle, thus disappears for high temperatures or for $\gamma\to 0$ and/or $\hbar\to 0$, as it should be. It is seen as well that the correction term has a sign opposite to the main effect: When increasing its mass, the particle absorbs heat, but the small correction in right-hand side of Eq. (4.7), which appears due to the common influence of the quantum effects and interaction with the bath, tends to make the effect smaller. This is consistent with previous observation that at low T the entropy decreases.

V. ON A POPULAR DERIVATION OF THE LANDAUER BOUND

Let us discuss in a more general perspective the obtained result on the violation of the Landauer principle. For this purpose we will analyze one of the simplest derivations of this principle [4–6], in order to understand what essentially goes into it and where its argument may be inapplicable.

The popular derivations go as follows. Erasure is accompanied by reduction of entropy of the information-carrying system. Since entropy of the overall system, which is the carrier plus bath, cannot decrease, one quickly concludes that entropy of the bath should increase, thereby producing heat. This argument seems to be rather solid, because, instead of involving any derivation, it just directly refers to the second law, namely, to the Clausius inequality. However, there are *three assumptions* in that inequality, which are rather restrictive. They immediately pertain to the Landauer inequality.

The first assumption is that the total entropy S of the overall system is sum of partial entropies of system and bath, $S=S_S+S_B$. This is obvious in classical systems, but may be invalid in quantum systems. The second is quick thermalization in the bath, implying $đ\mathcal{Q}_B=T\,dS_B$. The third assumption is smallness of the interaction energy $đ\mathcal{Q}_I$, allowing to conclude from energy conservation $đQ_S+đQ_B+đQ_I=0$ implies that $dS_B=đ\mathcal{Q}_B/T=-đ\mathcal{Q}_S/T$. With these assumptions it now follows immediately that $0\leqslant dS=dS_S+dS_B=dS_S-đ\mathcal{Q}_S/T$.

These asumptions are strictly valid only for noninteracting information carrier and its bath. However, without interaction there is no reason to speak about erasure. Under several additional conditions [18] these assumptions may be valid as certain *approximations* in the weak-coupling case. Their validity is especially endangered in the quantum regime where the complete entropy, which is the subject of the second law applied to the complete system, is not equal to the sum of the separate entropies if there occurs quantum entanglement. So the above simple derivation is actually restricted, as was noted already in the context of rather different physical arguments [26,27].

The general validity of the Landauer principle must be completely put on the validity of the Clausius inequality. It was our main message that quantum entanglement limits the validity of the Clausius inequality and, consequently, of the Landauer bound. It can be checked explicitly that in that regime the above assumptions are invalid: the interaction energy contributes to the total energy, and the total entropy is not the sum of two partial entropies [9,10]. Recently violations of other formulations of the second law were noticed and investigated in Refs. [28,29].

VI. CONCLUSION

The Landauer principle requires dissipation (release) of $T|dS|$ units of energy as a consequence of erasure of $|dS|$ units of information. This was believed to be the only *fundamental* energy cost of computational processes [4–6,8]. Though, in practice, computers dissipate much more energy, the Landauer principle was considered to put a general physical bound to which every computational device interacting with its thermal environment must satisfy. Indeed, in several physical situations the Landauer principle can be proved explicitly [8].

The main purpose of the present paper was to provide a counterexample of this principle, and thus to question its univeral validity. In the reported case all general requirements on the information carrier and its interaction with the bath are met. The only new point of our approach is that we were interested in sufficiently low temperatures, where quantum effects are relevant. The Landauer principle appeared to be violated by these effects (in particular, by entanglement). At high temperatures we reproduce its validity. In fact, in this limit our model is equivalent to that considered in Ref. [8], where the classical Landauer principle was derived in a quite general ground.

Recently the Landauer bound attracted a serious attention by workers in the field of applied information science [30]. There is a definite belief that this bound can be approached by further miniaturization of computational devices. It is hoped that the present paper will help to understand limitations of the Landauer principle itself, which may lead to unexpected mechanisms for computing in the quantum regime.

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[1] *Maxwell's Demon: Entropy, Information, Computing*, edited by H.S. Leff and A.F. Rex (Adam Hilger, London, 1990).

[2] J. von Neumann, *Theory of Self-Reproducing Automata*, Lecture 3 (University Illinois Press, Urbana, IL, 1966).

[3] L. Szilard, Z. Phys. **53**, 840 (1929), reprinted in [1].

[4] R. Landauer, IBM J. Res. Dev. **5**, 183 (1961), reprinted in [1].

[5] R. Landauer, Nature (London) **355**, 779 (1988).

[6] C.H. Bennett, Int. J. Theor. Phys. **21**, 905 (1982), reprinted in [1].

[7] W.H. Zurek, Nature (London) **347**, 119 (1989).

[8] K. Shizume, Phys. Rev. E **52**, 3495 (1995).

[9] A.E. Allahverdyan and Th.M. Nieuwenhuizen, Phys. Rev. Lett. **85**, 1799 (2000).

[10] Th.M. Nieuwenhuizen and A.E. Allahverdyan, e-print cond-mat/0011389; A.E. Allahverdyan and Th.M. Nieuwenhuizen (unpublished).

[11] Th.M. Nieuwenhuizen, Phys. Rev. Lett. **80**, 5580 (1998); Phys. Rev. E **61**, 267 (2000).

[12] A.E. Allahverdyan and Th.M. Nieuwenhuizen, Phys. Rev. E **62**, 845 (2000).

[13] I. Chisar and Y. Kerner, *Information Theory* (Academic Press, New York, 1983).

[14] T.N. Cover and J.A. Thomas, *Elements of Information Theory* (Academic Press, New York, 1983).

[15] E. Jaynes, Phys. Rev. **108**, 171 (1957); **108**, 620 (1957); Am. J. Phys. **31**, 66 (1963).

[16] A.N. Kolmogorov, Probl. Inf. Transm. **1**, 1 (1965).

[17] The literature devoted to the correct interpretation of heat and work is broad. We cite only the well-known books: J. Keizer, *Statistical Thermodynamics of Nonequilibrium Processes* (Springer-Verlag, Berlin, 1987); Yu. L. Klimontovich, *Statistical Theory of Open Systems* (Kluwer, Amsterdam, 1997).

[18] R. Balian, *From Microphysics to Macrophysics* (Springer-Verlag, 1992), Vols. I and II.

[19] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993).

[20] A. Caldeira and A. Leggett, Phys. Rev. A **31**, 1059 (1985).

[21] G.W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965).

[22] P. Ullersma, Physica (Amsterdam) **32**, 27 (1966); **32**, 56 (1966); **32**, 74 (1966); **32**, 90 (1966).

[23] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1984).

[24] F. Haake and R. Reibold, Phys. Rev. A **32**, 2462 (1985).

[25] H. Grabert, U. Weiss, and P. Talkner, Z. Phys. B: Condens. Matter **55**, 87 (1984).

[26] J. Berger, Int. J. Theor. Phys. **29**, 985 (1990).

[27] D. Wolpert, Phys. Today **45**, 98 (1992).

[28] V. Capek and J. Bok, Czech. J. Phys. **49**, 1645 (1999); e-print cond-mat/9905232.

[29] V. Capek, e-print cond-mat/0012056.

[30] R.C. Merkle, Nanotechnology **4**, 21 (1993); **4**, 114 (1993).